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# Validity of the phase approximation for coupled nonlinear oscillators: a case study\*

Alessio Franci<sup>1</sup>, William Pasillas-Lépine<sup>2</sup>, and Antoine Chaillet<sup>3</sup>

**Abstract**—Motivated by neuroscience applications, we rigorously derive the phase dynamics of an ensemble of interconnected nonlinear oscillators under the effect of a proportional feedback. We individuate the critical parameters determining the validity of the phase approximation and derive bounds on the accuracy of the latter in reproducing the behavior of the original system. We use these results to study the existence of oscillating phase-locked solutions in the original oscillator model.

## I. INTRODUCTION

The use of the phase dynamics associated to nonlinear oscillators is a widely accepted tool to rigorously analyze complex collective phenomena like synchronization, pattern formation, and resonance. Examples of such behaviors are found, for instance, in biology [1], physics [2], and engineering [3], [4], [5], [6], [7], [8]. However, the reduction of periodic oscillatory dynamics to the associated phase model is relevant only if the inputs and disturbances are small compared to the attractivity of the limit cycle. This problem is of crucial importance in control engineering applications, where inputs plays a fundamental role and some performance or security criteria have to be satisfied.

Motivated by neuroscience application, we have recently developed a feedback control law that aims at altering the synchronization in an interconnected neuronal population. Under some assumptions, the phase dynamics of the closed-loop system was analytically computed and sufficient conditions for different control objectives were derived [9], [10]. Nevertheless, the relevance of these results for the original ensemble of nonlinear oscillators (modeling the neuronal population) is not straightforward due to aforementioned intrinsic limitations of the phase reduction.

In this paper, we generalize the oscillator model introduced in [9] and study its phase dynamics. This generalization permits to describe a wider range of coupling and feedback schemes and embraces in a unified model different interesting special cases. Based on classical results on normal hyperbolic invariant manifolds, we rigorously derive the closed-loop phase dynamics and individuate the parameters determining its validity, along with explicit bounds on its

accuracy in reproducing the behavior of the original system. These results are used to study the existence of phase-locked solutions in the original nonlinear oscillator population. More precisely, we show that, if the coupling and feedback strengths are sufficiently small, then generically no phase-locked solutions oscillating with non-zero frequency can exist.

## Notation and preliminaries

$\mathbb{R}_{\geq 0}^n$  denotes the closed orthant  $\{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ , whereas  $\mathbb{R}_{> 0}^n$  denotes the open orthant  $\{x \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$ .  $T^n$  denotes the  $n$ -torus.

**Norms:** Given  $n, m \in \mathbb{N}$  and  $A = \{A_{ij}\}_{i=1, \dots, n, j=1, \dots, m} \in \mathbb{R}^{n \times m}$ , we denote the Frobenius norm of  $A$  as  $|A| := \sqrt{\sum_{i=1}^n \sum_{j=1}^m A_{ij}^2}$ . When either  $n = 1$  or  $m = 1$ ,  $|\cdot|$  is the Euclidean norm.

**Splittings:** Given a finite dimensional vector space  $V$ , a splitting of  $V$  is a collection of linear subspace  $V_i \subset V$ ,  $i = 1, \dots, l$ , such that  $V = \bigoplus_{i=1}^l V_i$ , where  $\bigoplus$  denotes the direct sum. Given a linear application  $A : V \rightarrow V$ , an  $A$ -invariant splitting is a splitting  $V = \bigoplus_{i=1}^l V_i$  such that  $AV_i \subset V_i$ ,  $i = 1, \dots, l$ .

**Tangent maps:** Given a  $n$ -dimensional manifold  $\mathcal{M}$ , we denote its tangent space at  $x \in \mathcal{M}$  as  $T_x \mathcal{M}$ , and similarly for submanifolds. Given a set  $W \subset \mathcal{M}$  and a map  $f : \mathcal{M} \rightarrow \mathbb{R}^m$ , we denote by  $f|_W : W \rightarrow \mathbb{R}^m$  the restriction of  $f$  to  $W$ , i.e.  $f|_W(x) = f(x)$  for all  $x \in W$ . The tangent application (or differential, or linearization) of a  $C^1$  function  $f : \mathcal{M} \rightarrow \mathbb{R}^m$  is denoted as  $Df$ , i.e. in coordinate  $Df(x) = \frac{\partial f}{\partial x}(x)$ .

**Measure:** The Lebesgue measure on  $\mathbb{R}^n$  is denoted by  $\mu$ , and for almost all ( $\forall a.a.$ ) denotes the equivalence operation with respect to this measure.

## II. LANDAU-STUART OSCILLATORS WITH DIFFUSIVE AND FEEDBACK COUPLING

We start by introducing the coupled oscillator system under analysis. Given  $\rho_i > 0$ ,  $i = 1, \dots, N$ , consider the following dynamics on  $\mathbb{C}^N$

$$\dot{z}_i = (i\omega_i + \rho_i^2 - |z_i|^2)z_i + \sum_{j=1}^N \kappa_{ij} e^{i\delta_{ij}} (e^{i\eta_j} z_j - e^{i\eta_i} z_i) + u_i, \quad (1)$$

where  $\kappa := [\kappa_{ij}]_{i,j=1, \dots, N} \in \mathbb{R}^{N \times N}$ ,

$$u_i := \sum_{j=1}^N \tilde{\gamma}_{ij} e^{i\phi_{ij}} [\cos \varphi_j \operatorname{Re}(e^{i\psi_j} z_j) + i \sin \varphi_j \operatorname{Im}(e^{i\psi_j} z_j)], \quad (2)$$

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and  $\tilde{\gamma} := [\tilde{\gamma}_{ij}]_{i,j=1,\dots,N} \in \mathbb{R}^{N \times N}$ . We also define

$$\Phi := ([\delta_{ij}]_{i,j=1,\dots,N}, [\eta_i]_{i=1,\dots,N}, [\varphi_i]_{i=1,\dots,N}, [\phi_{ij}]_{i,j=1,\dots,N}, [\psi_i]_{i=1,\dots,N}) \in \mathbb{R}^{N \times (2N+3)} \quad (3)$$

The dynamics (1) can be split in three parts. The term  $(i\omega_i + \rho_i^2 - |z_i|^2)z_i$  is the oscillator internal dynamics. It corresponds to a stable oscillation of radius  $\rho_i$  and frequency  $\omega_i$  and is commonly referred to as Landau-Stuart oscillator, which represents a normal form of a supercritical Andronov-Hopf bifurcation [11]. The second term constitutes a linear coupling between the oscillators, where  $\kappa_{ij}e^{i\delta_{ij}}$  is the (complex) coupling gain and where the phases  $\eta_i$  rotate the oscillator contribution to the coupling. The last term  $u_i$  constitutes a feedback coupling term which injects the output of each oscillator  $y_j := [\cos \varphi_j \text{Re}(e^{i\psi_j} z_j) + i \sin \varphi_j \text{Im}(e^{i\psi_j} z_j)]$  back in the network with complex gains  $B_{ij} := \tilde{\gamma}_{ij}e^{i\phi_{ij}}$ .

The analysis of (1) is motivated by the two following special cases:

#### Special case 1

The choice  $\Phi = 0$  and  $\tilde{\gamma}_{ij} = \beta_i \alpha_j$ ,  $i, j = 1, \dots, N$ , for some  $\alpha := [\alpha_i]_{i=1,\dots,N} \in \mathbb{R}^N$  and  $\beta := [\beta_i]_{i=1,\dots,N} \in \mathbb{R}^N$ , reduces (1) to

$$\begin{aligned} \dot{z}_i &= (i\omega_i + \rho_i^2 - |z_i|^2)z_i + \sum_{j=1}^N \kappa_{ij}(z_i - z_j) + \beta_i y, \\ y &:= \sum_{j=1}^N \alpha_j \text{Re}(z_j). \end{aligned}$$

This dynamics constitutes an ensemble of diffusively coupled Landau-Stuart oscillators under the effect of proportional mean-field feedback, where  $\beta$  is the feedback gain and  $y$  is the ensemble mean-field. Motivated by neuroscience applications, this model was recently used in [9], [10] to analyze the behavior of an ensemble of diffusively coupled periodically spiking neurons under the effect of an electrical stimulation that is proportional to the ensemble mean membrane voltage. In those works, the membrane voltage of each neuron is represented by the real part of the associated oscillator, whereas the imaginary part of the oscillation accounts for the effects of other physical variables. More modeling details can be found in [9].

The introduction of the phases  $\Phi$  in (1) accounts for possible imprecision in the association between physical (voltages, conductances, ion concentrations, etc.) and mathematical (real and imaginary parts) variables. For instance:

- The phases  $[\eta_i]_{i=1,\dots,N}$  rotate the oscillator contributions to the diffusive coupling. This permits to consider the case when, in the simplification from the full coupled neuronal limit cycles to the reduced ones, we can not exactly associate the voltages and the other physical variables of each oscillator to the real and imaginary parts, respectively.
- The phases  $[\delta_{ij}]_{i,j=1,\dots,N}$  rotate the diffusive coupling terms in such a way that the imaginary part of the coupling influences the real one and vice-versa.
- Similarly, the phases  $[\varphi_i]_{i=1,\dots,N}$ ,  $[\phi_{ij}]_{i,j=1,\dots,N}$ , and  $[\psi_i]_{i=1,\dots,N}$  accounts for the same type of inaccuracies

in the feedback coupling. In particular, the phases  $[\varphi_i]_{i=1,\dots,N}$  permit to consider the case when the real and the imaginary parts of the oscillations contribute to the mean-field measurement with different gains.

#### Special case 2

Another interesting special case of (1) is given by the normal form (A.4) in [12], which constitutes the basis of many theoretical works on synchronization phenomena between coupled oscillators [13], [14], [15], just to name a few examples. Indeed, with the choice  $\varphi_i = \frac{\pi}{4}$ ,  $\eta_i = \psi_i = 0$ , and  $\phi_{ij} = \delta_{ij}$ , for all  $i, j = 1, \dots, N$ , the sum of the coupling and feedback terms in Equations (1)-(2) can be rewritten as

$$\begin{aligned} &\sum_{j=1}^N e^{i\delta_{ij}} [(\kappa_{ij} + \tilde{\gamma}_{ij})z_j - \kappa_{ij}z_i] \\ &= \sum_{j=1}^N (\kappa_{ij} + \tilde{\gamma}_{ij})e^{i\delta_{ij}} \left[ z_j - \frac{\kappa_{ij}}{\kappa_{ij} + \tilde{\gamma}_{ij}} z_i \right]. \end{aligned} \quad (4)$$

Clearly, for  $\kappa_{ij} = 0$ , we obtain a purely direct coupling, which recovers the coupling term in Equation (A.4) of [12] with  $\tilde{\kappa}_i = 0$ . Otherwise, given  $\tilde{\kappa}_i \in (0, 1]$ , we let

$$\frac{\kappa_{ij}}{\kappa_{ij} + \tilde{\gamma}_{ij}} = \tilde{\kappa}_i,$$

which is equivalent to asking

$$\tilde{\gamma}_{ij} = \frac{(1 - \tilde{\kappa}_i)\kappa_{ij}}{\tilde{\kappa}_i}. \quad (5)$$

By plugging (5) into (4), we can further transform the coupling and feedback terms as

$$\sum_{j=1}^N \left( \kappa_{ij} + \frac{1 - \tilde{\kappa}_i}{\tilde{\kappa}_i} \kappa_{ij} \right) e^{i\delta_{ij}} (z_j - \tilde{\kappa}_i z_i) = \sum_{j=1}^N \kappa'_{ij} e^{i\delta_{ij}} (z_j - \tilde{\kappa}_i z_i),$$

where  $\kappa'_{ij} := \frac{\kappa_{ij}}{\tilde{\kappa}_i}$ , which recovers the coupling term of Equation (A.4) of [12] with  $\tilde{\kappa}_i \in (0, 1]$ .

### III. FORMAL REDUCTION TO THE PHASE DYNAMICS

The goal of this section is to derive the phase dynamics of the closed-loop system (1)-(2). We start by writing the oscillator states in polar coordinates, that is  $z_i =: r_i e^{i\theta_i}$ , for all  $i = 1, \dots, N$ , where  $r_i = |z_i| \in \mathbb{R}_{\geq 0}$  and  $\theta_i = \arg(z_i) \in T^1$ . We stress that the oscillator phases  $\theta_i$  are defined only for  $|z_i| = r_i > 0$ . In these coordinates the dynamics (1)-(2) reads

$$\begin{aligned} \dot{r}_i e^{i\theta_i} + i r_i \dot{\theta}_i e^{i\theta_i} &= (i\omega_i + \rho_i^2 - r_i^2) r_i e^{i\theta_i} + \\ &\sum_{j=1}^N \kappa_{ij} e^{i\delta_{ij}} (e^{i\eta_j} r_j e^{i\theta_j} - e^{i\eta_i} r_i e^{i\theta_i}) + u_i \end{aligned}$$

where

$$u_i = \sum_{j=1}^N \tilde{\gamma}_{ij} e^{i\phi_{ij}} \left[ \cos \varphi_j \text{Re}(e^{i\psi_j} r_j e^{i\theta_j}) + i \sin \varphi_j \text{Im}(e^{i\psi_j} r_j e^{i\theta_j}) \right].$$

By multiplying both sides of this dynamics by  $\frac{e^{-i\theta_i}}{r_i}$ , extracting the real and imaginary part, and using some basic trigonometry, we get, for  $r_i > 0$ ,  $i = 1, \dots, N$ ,

$$\dot{\theta}_i = \omega_i + f_i(\theta, r, \kappa, \tilde{\gamma}, \Phi) \quad (6a)$$

$$\dot{r}_i = r_i(\rho_i^2 - r_i^2) + g_i(\theta, r, \kappa, \tilde{\gamma}, \Phi), \quad (6b)$$

where, for all  $i = 1, \dots, N$ ,

$$\begin{aligned} f_i(\theta, r, \kappa, \tilde{\gamma}, \Phi) &:= \\ & - \sum_{j=1}^N \kappa_{ij} \sin(\delta_{ij} + \eta_i) + \sum_{j=1}^N \frac{\kappa_{ij} r_j}{r_i} \sin(\theta_j - \theta_i + \delta_{ij} + \eta_j) \\ & + \sum_{j=1}^N \frac{\tilde{\gamma}_{ij} r_j}{r_i} \left[ \frac{\sin \varphi_j + \cos \varphi_j}{2} \sin(\theta_j - \theta_i + \phi_{ij} + \psi_j) \right. \\ & \quad \left. + \frac{\sin \varphi_j - \cos \varphi_j}{2} \sin(\theta_j + \theta_i - \phi_{ij} + \psi_j) \right] \\ g_i(\theta, r, \kappa, \tilde{\gamma}, \Phi) &:= \\ & - r_i \sum_{j=1}^N \kappa_{ij} \cos(\delta_{ij} + \eta_i) + \sum_{j=1}^N \kappa_{ij} r_j \cos(\theta_j - \theta_i + \delta_{ij} + \eta_j) \\ & + \sum_{j=1}^N \tilde{\gamma}_{ij} r_j \left[ \frac{\sin \varphi_j + \cos \varphi_j}{2} \cos(\theta_j - \theta_i + \phi_{ij} + \psi_j) \right. \\ & \quad \left. + \frac{\cos \varphi_j - \sin \varphi_j}{2} \cos(\theta_j + \theta_i - \phi_{ij} + \psi_j) \right] \end{aligned}$$

which defines the phase/radius dynamics of (1)-(2) on  $T^N \times \mathbb{R}_{>0}^N$ .

Let  $f := [f_i]_{i=1, \dots, N}$  and  $g := [g_i]_{i=1, \dots, N}$ . When the diffusive coupling and the feedback are both zero, that is  $\kappa = \tilde{\gamma} = 0$ , we have  $f \equiv g \equiv 0$ . Hence, in this case, equation (6) reduces to

$$\begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} = H(\theta, r) := \begin{bmatrix} \omega \\ r(\rho^2 - r^2) \end{bmatrix} \quad (7)$$

where  $\rho := [\rho_i]_{i=1, \dots, N} \in \mathbb{R}^N$ . It is obvious that the  $N$ -torus  $\mathcal{T}_0 := T^N \times \{\rho\} \subset T^N \times \mathbb{R}_{>0}^N$  is invariant for (7), since all its points are fixed points of the radius dynamics in (7). Moreover, it is normally hyperbolic as defined and proved below.

Given an  $n$ -dimensional smooth Riemannian manifold  $\mathcal{M}$ , with metric  $\langle \cdot, \cdot \rangle_R$ , the solution of an autonomous dynamical system

$$\dot{x} = F(x), \quad x \in \mathcal{M}, \quad (8)$$

starting at  $x_0 \in \mathcal{M}$  at  $t = 0$  is denoted as  $x(\cdot, x_0)$  everywhere it exists. Let  $\|\cdot\|_R$  be the norm induced by the Riemannian metric and let  $\mathcal{N} \subset \mathcal{M}$  be a smooth compact  $m$ -dimensional submanifold. We define *normal hyperbolicity* of (8) at  $\mathcal{N}$  as follows [16], [17].

**Definition 1.** The dynamical system (8) is *normally hyperbolic* at  $\mathcal{N}$  if the two following conditions are satisfied:

i) For all  $x \in \mathcal{N}$ , there exists a  $DF_x$ -invariant splitting

$$T_x \mathcal{M} = N_x^u \oplus T_x \mathcal{N} \oplus N_x^s$$

of  $T_x \mathcal{M}$  over  $\mathcal{N}$ . In this case, for all  $x \in \mathcal{N}$ , denote  $DF_x^u := DF(x)|_{N_x^u}$ ,  $DF_x^0 := DF(x)|_{T_x \mathcal{N}}$ , and  $DF_x^s := DF(x)|_{N_x^s}$ .

ii) We have either  $N_x^r = \emptyset$ , for all  $x \in \mathcal{N}$ ,  $r = s, u$ , or:

$$ii-a) \inf_{x \in \mathcal{N}} \inf_{v \in N_x^u} \frac{|e^{DF_x^u} v|_R}{|v|_R} > \max \left\{ 1, \sup_{x \in \mathcal{N}} \sup_{v \in T_x \mathcal{N}} \frac{|e^{DF_x^0} v|_R}{|v|_R} \right\}$$

$$ii-b) \sup_{x \in \mathcal{N}} \sup_{v \in N_x^s} \frac{|e^{DF_x^s} v|_R}{|v|_R} < \min \left\{ 1, \inf_{x \in \mathcal{N}} \inf_{v \in T_x \mathcal{N}} \frac{|e^{DF_x^0} v|_R}{|v|_R} \right\}$$

The two subspaces  $N_x^u$  and  $N_x^s$  are called the unstable and the stable space of  $T_x \mathcal{N}$  with respect to the tangent application of (8), respectively.

Conditions *ii-a)* (resp. *ii-b)*) means that, when  $N_x^u \neq \emptyset$  (resp.  $N_x^s \neq \emptyset$ ), the flow of (8) expands (resp. contracts) the unstable (resp. stable) space more sharply than its tangent space.

**Lemma 1.** The dynamics (7) is normally hyperbolic at  $\mathcal{T}_0$ .

*Proof.* Denote  $\mathcal{Q} := T^N \times \mathbb{R}_{>0}^N$ . Moreover, denote  $\vartheta := (\theta, \rho) \in \mathcal{T}_0$ . Since  $r$  is constant on  $\mathcal{T}_0$ , the tangent space  $T_\vartheta \mathcal{T}_0$  at a point  $\vartheta \in \mathcal{T}_0$  is spanned by  $\left\{ \frac{\partial}{\partial \theta_i} \Big|_\vartheta, i = 1, \dots, N \right\}$ . Let

$$N_\vartheta^s := \text{span} \left\{ \frac{\partial}{\partial r_i} \Big|_\vartheta, i = 1, \dots, N \right\}.$$

Fix coordinates in the tangent space in such a way that  $(\hat{e}^i, 0_N) = \frac{\partial}{\partial \theta_i}$ ,  $i = 1, \dots, N$ , is a base of  $T_\vartheta \mathcal{T}_0$ , where  $\hat{e}_j^i = 0$  if  $j \neq i$  and  $\hat{e}_i^i = 1$ , and, similarly,  $(0_N, \hat{e}^i) = \frac{\partial}{\partial r_i}$ ,  $i = 1, \dots, N$ , is a base of  $N_\vartheta^s$ . In these bases, for all  $\vartheta \in \mathcal{T}_0$  the tangent application of (7) at  $\vartheta$  reads

$$\begin{aligned} DH_\vartheta &= \begin{bmatrix} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial r} \\ \frac{\partial \dot{r}}{\partial \theta} & \frac{\partial \dot{r}}{\partial r} \end{bmatrix} \Big|_\vartheta \\ &= \begin{bmatrix} 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & \text{diag}\{-2\rho_i^2\}_{i=1, \dots, N} \end{bmatrix}. \end{aligned} \quad (9)$$

It follows from (9) that, for all  $\vartheta \in \mathcal{T}_0$  the splitting

$$T_\vartheta \mathcal{Q} = T_\vartheta \mathcal{T}_0 \oplus N_\vartheta^s$$

is  $DH_\vartheta$ -invariant, which verifies condition *i)* of Definition 1 with  $N_{(\theta, \rho)}^u = \emptyset$ . In particular, in the bases  $(\hat{e}^i, 0_N)$  and  $(0_N, \hat{e}^i)$ , we have

$$DH_\vartheta^0 := DH_{(\theta, \rho)}|_{T_\vartheta \mathcal{T}_0} = 0_{2N \times 2N} \quad (10a)$$

$$\begin{aligned} DH_\vartheta^s &:= DH_\vartheta|_{N_\vartheta^s} \\ &= \begin{bmatrix} 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & \text{diag}\{-2\rho_i^2\}_{i=1, \dots, N} \end{bmatrix}. \end{aligned} \quad (10b)$$

We now endow  $\mathcal{Q}$  with a suitable Riemannian metric. To this aim note that since  $D_\rho := \text{diag}\{-2\rho_i^2\}_{i=1, \dots, N}$  is Hurwitz, there exists a positive definite matrix  $P \in \mathbb{R}^{N \times N} > 0$  such that the exponential application  $e^{D_\rho}$  contracts the norm  $|x|_P := \sqrt{x^T P x}$  induced by  $P$ , that is  $|e^{D_\rho} x|_P < |x|_P$

for all  $x \in \mathbb{R}^N$ . Since  $D_\rho$  does not depend on  $\vartheta$ , so does  $P$ . Let

$$\tilde{P} := \begin{bmatrix} I_N & 0_{N \times N} \\ 0_{N \times N} & P \end{bmatrix}.$$

We endow  $\mathcal{Q}$  with the constant Riemannian metric  $\langle \cdot, \cdot \rangle_{\tilde{P}}$  defined by

$$\langle v, u \rangle_{\tilde{P}} = v^T \tilde{P} u, \quad v, u \in T_{(\theta, r)} \mathcal{Q}.$$

In the base  $(O_N, \hat{e}^i)$ ,  $i = 1, \dots, N$ , a generic vector  $v \in N_\vartheta^s$  is represented by  $(0_N, v^r)$ ,  $v^r \in \mathbb{R}^N$ . Therefore, we have

$$\begin{aligned} \sup_{\vartheta \in \mathcal{T}_0} \sup_{v \in N_\vartheta^s} \frac{|e^{DH_\vartheta^s} v|_{\tilde{P}}}{|v|_{\tilde{P}}} &= \sup_{\vartheta \in \mathcal{T}_0} \sup_{v^r \in \mathbb{R}^N} \frac{|e^{DH_\vartheta^s}(0_N, v^r)|_{\tilde{P}}}{|(0, v^r)|_{\tilde{P}}} \\ &= \sup_{\vartheta \in \mathcal{T}_0} \sup_{v^r \in \mathbb{R}^N} \frac{|e^{D_\rho} v^r|_P}{|v^r|_P} < 1. \end{aligned}$$

Condition *ii-b*) of Definition 1 follows by noticing that, since  $DH_\vartheta^0 = 0_{2N \times 2N}$ ,

$$\min \left\{ 1, \inf_{\vartheta \in \mathcal{T}_0} \inf_{v \in T_\vartheta \mathcal{T}_0} \frac{|e^{DH_\vartheta^0} v|_{\tilde{P}}}{|v|_{\tilde{P}}} \right\} = 1.$$

□

We are now going to apply a classical result of Hirsch et. al [16, Theorem 4.1] to show that, if  $\kappa, \tilde{\gamma}$  are sufficiently small, then (6) still has an attractive normally hyperbolic invariant manifold in a neighborhood of  $\mathcal{T}_0$ .

**Theorem 1.** *Given  $\rho_i > 0$ ,  $i = 1, \dots, N$ , there exists constants  $\delta_h, C_h > 0$  depending only  $\rho_i$ ,  $i = 1, \dots, N$ , such that, if*

$$|(\kappa, \tilde{\gamma})| < \delta_h \quad (11)$$

*then there exists an attractive invariant manifold  $\mathcal{T}_p \subset T^N \times \mathbb{R}_{>0}^N$  normally hyperbolic for (6) and satisfying*

$$|r - \rho| \leq C_h |(\kappa, \tilde{\gamma})|, \quad \forall (\theta, r) \in \mathcal{T}_p. \quad (12)$$

Theorem 1 states that, if the coupling and the feedback strengths are smaller than a constant  $\delta_h$  depending only on the natural radius  $\rho_i$ , then the network dynamics (1) evolves on an (attractive) normally hyperbolic invariant manifold  $\mathcal{T}_p$ . Moreover, the distance between  $\mathcal{T}_p$  and  $\mathcal{T}_0$  is less than  $C_h |(\kappa, \tilde{\gamma})|$ , where again  $C_h$  depends only on the natural radius. We refer to condition (11) as the *small coupling condition*.

**Remark 1.** Note that the constant  $\delta_h$ , and thus the small coupling condition, depends only on the oscillator natural radius  $\rho_i$ . In particular, it is independent of the natural frequencies  $\omega$ .

If the small coupling condition is satisfied, then Theorem 1 has two important consequences:

1. On the attractive normally hyperbolic invariant torus  $\mathcal{T}_p$ , the oscillator radius variations around their natural radius are bounded by  $|r(t) - \rho| \leq C_h |(\kappa, \tilde{\gamma})|$ , for all  $t \geq 0$ . In particular, they are small, provided that  $|(\kappa, \tilde{\gamma})|$  is small.

2. To the first order in  $|(\kappa, \tilde{\gamma})|$  the phase dynamics does not depend on the radius dynamics. Indeed from (12) and (18) it follows that

$$\begin{aligned} \left| \frac{\partial f}{\partial r}(\theta, r, \kappa, \tilde{\gamma}, \Phi)(r - \rho) \right| &\leq \left| \frac{\partial f}{\partial r}(\theta, r, \kappa, \tilde{\gamma}, \Phi) \right| |r - \rho| \\ &\leq C_f |(\kappa, \tilde{\gamma})| C_h |(\kappa, \tilde{\gamma})| \\ &= C_f C_h |(\kappa, \tilde{\gamma})|^2. \end{aligned}$$

Hence, to the first order in  $|(\kappa, \tilde{\gamma})|$ , *i.e.* to the first order in the coupling and feedback strength, if the small coupling condition (11) holds true, then (6) boils down to the *phase dynamics* equation

$$\dot{\theta}_i = \omega_i + \bar{f}_i(\theta, \kappa, \tilde{\gamma}, \Phi), \quad (13)$$

where

$$\begin{aligned} \bar{f}_i(\theta, \kappa, \tilde{\gamma}, \Phi) &:= \\ &= - \sum_{j=1}^N \kappa_{ij} \sin(\delta_{ij} + \eta_i) + \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i + \delta_{ij} + \eta_j) \\ &+ \sum_{j=1}^N \gamma_{ij} \left[ \frac{\sin \varphi_j + \cos \varphi_j}{2} \sin(\theta_j - \theta_i + \phi_{ij} + \psi_j) \right. \\ &\quad \left. + \frac{\sin \varphi_j - \cos \varphi_j}{2} \sin(\theta_j + \theta_i - \phi_{ij} + \psi_j) \right], \end{aligned} \quad (14)$$

with  $k := [k_{ij}]_{ij=1, \dots, N} := \left[ \frac{\kappa_{ij} \rho_j}{\rho_i} \right]$  and  $\gamma := [\gamma_{ij}]_{ij=1, \dots, N} := \left[ \frac{\tilde{\gamma}_{ij} \rho_j}{\rho_i} \right]$ , and the radius dynamics can be neglected, as it was done, for instance, in [18], [9].

**Remark 2.** We stress that, if (11) is satisfied, the error between the nominal dynamics (6) and its phase dynamics (13) is of the same order as  $|(\kappa, \tilde{\gamma})|^2$ .

*Proof.* Even though for  $\kappa = \tilde{\gamma} = 0$  it holds that  $f \equiv g \equiv 0$ , as soon as  $(\kappa, \tilde{\gamma}) \neq 0$ ,  $f$  and  $g$  are unbounded, due to singularities at  $r_i = 0$  and  $r_i = \infty$ ,  $i = 1, \dots, N$ . However, the persistence of the normally hyperbolic invariant torus solely relying on local arguments, we can construct a locally defined auxiliary smooth dynamical system, which is identical to (6) near  $\mathcal{T}_0$ . The auxiliary system possesses a normally hyperbolic invariant manifold  $\mathcal{T}_p$  near  $\mathcal{T}_0$  if and only if the same holds for the original dynamics (6).

STEP 1: *Compactification.*

The result of [16, Theorem 4.1] applies for dynamical systems defined on compact manifolds. Thus, we construct our auxiliary dynamics on a compact manifold containing  $\mathcal{T}_0$ . To this end, we consider some smooth functions  $G_i : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$  such that (see [19, Page 54])

$$G_i(r_i) = \begin{cases} 0 & \text{if } r_i \in [0, \frac{\rho_i}{2}] \\ 1 & \text{if } r_i \in [\frac{3\rho_i}{4}, \frac{5\rho_i}{4}] \\ 0 & \text{if } r_i \geq \frac{3\rho_i}{2}. \end{cases} \quad (15)$$

By denoting

$$\bar{P} := \left\{ r \in \mathbb{R}_{>0}^N : r_i \in \left[ \frac{\rho_i}{2}, \frac{3\rho_i}{2} \right], i = 1, \dots, N \right\},$$

we let  $\overline{\mathcal{M}}$  be the compact submanifold

$$\overline{\mathcal{M}} := T^N \times \overline{P}.$$

We define our auxiliary dynamics as a dynamical system on the compact submanifold  $\overline{\mathcal{M}}$  as follows:

$$\dot{\theta}_i = \omega_i + f_i(\theta, r, \kappa, \tilde{\gamma}, \Phi), \quad \theta \in T^N \quad (16a)$$

$$\dot{r}_i = G_i(r_i) \left( r_i(\rho_i^2 - r_i^2) + g_i(\theta, r, \kappa, \tilde{\gamma}, \Phi) \right), \quad r \in \overline{P}. \quad (16b)$$

Note that, by definition, the two dynamics (6) and (16) coincide on

$$\underline{\mathcal{M}} := T^N \times \underline{P}, \quad (17)$$

where

$$\underline{P} := \left\{ r \in \mathbb{R}_{>0}^N : r_i \in \left[ \frac{3\rho_i}{4}, \frac{5\rho_i}{4} \right], \quad i = 1, \dots, N \right\}.$$

Hence, (6) has an attractive normally hyperbolic invariant manifold  $\mathcal{T}_p \subset \underline{\mathcal{M}}$  if and only if (16) does.

STEP 2: *Invariance.*

In order to apply [16, Theorem 4.1] to (16) with  $\kappa$  and  $\tilde{\gamma}$  as the perturbation parameters, we also have to show that the compact manifold  $\overline{\mathcal{M}}$  is invariant with respect to the flow of (16) independently of  $(\kappa, \tilde{\gamma}) \in \mathbb{R}^{N \times 2N}$ . In this case, for all perturbation parameters, the flow associated to (16) defines a diffeomorphism of  $\overline{\mathcal{M}}$ , as required by [16, Theorem 4.1]. By construction of the functions  $G_i$  in (15), the border of  $\overline{\mathcal{M}}$ , i.e.

$$\partial \overline{\mathcal{M}} := \left( T^N \times \left\{ \frac{\rho}{2} \right\} \right) \cup \left( T^N \times \left\{ \frac{3\rho}{2} \right\} \right)$$

is made of fixed points of the radius dynamics (16b), independently of the value of the parameters  $\kappa, \tilde{\gamma}, \Phi$ . In other words, for all  $(\kappa, \tilde{\gamma}, \Phi) \in \mathbb{R}^{N \times (4N+3)}$ , the border of  $\overline{\mathcal{M}}$  is given by the union of the two invariant torus  $T^N \times \left\{ \frac{\rho}{2} \right\}$  and  $T^N \times \left\{ \frac{3\rho}{2} \right\}$ . This in turn ensures that  $\overline{\mathcal{M}}$  is invariant for (16). To see this, suppose  $\overline{\mathcal{M}}$  is not invariant. Then, by continuity of the solutions of (16) there must exist some initial conditions  $(\theta_0, r_0) \in \overline{\mathcal{M}}$ , an instant  $\bar{t} \in \mathbb{R}$ , some  $\epsilon > 0$ , and a trajectory  $(\theta(\cdot, (\theta_0, r_0)), r(\cdot, (\theta_0, r_0)))$  of (16), such that  $(\theta(\bar{t}, (\theta_0, r_0)), r(\bar{t}, (\theta_0, r_0))) \in \partial \overline{\mathcal{M}}$  and  $(\theta(\bar{t} + \epsilon, (\theta_0, r_0)), r(\bar{t} + \epsilon, (\theta_0, r_0))) \notin \overline{\mathcal{M}}$ , which violates the invariance of  $\partial \overline{\mathcal{M}}$ .

STEP 3: *The nominal invariant manifold and construction of the perturbed one.*

For  $\kappa = \tilde{\gamma} = 0$ , the N-torus  $\mathcal{T}_0 = T^N \times \{\rho\}$  is attractive normally hyperbolic invariant for (16), since  $\mathcal{T}_0 \subset \underline{\mathcal{M}}$  and the same holds for (6).

It remains to show that, if  $|(\kappa, \tilde{\gamma})|$  is small, then the  $C^1$ -norm<sup>1</sup> of the functions

$$\begin{aligned} \overline{\mathcal{M}} &\longrightarrow \mathbb{R}^N \\ (\theta, r) &\longmapsto f|_{\overline{\mathcal{M}}}(\theta, r, \kappa, \tilde{\gamma}, \Phi) \end{aligned}$$

<sup>1</sup>The  $C^1$ -norm of a  $C^1$  bounded function with bounded derivatives  $F : \overline{\mathcal{M}} \rightarrow \mathbb{R}^N$  is defined as  $\|F\|_1 := \max\{\sup_{x \in \overline{\mathcal{M}}} |F(x)|, \sup_{x \in \overline{\mathcal{M}}} |\partial F / \partial x(x)|\}$ , where  $|F(x)|$  (resp.  $|\partial F / \partial x(x)|$ ) is the Euclidean (resp. Frobenius) norm of  $F(x)$  (resp.  $\partial F / \partial x(x)$ ).

and

$$\begin{aligned} \overline{\mathcal{M}} &\longrightarrow \mathbb{R}^N \\ (\theta, r) &\longmapsto G|_{\overline{\mathcal{M}}}(r)g|_{\overline{\mathcal{M}}}(\theta, r, \kappa, \tilde{\gamma}, \Phi) \end{aligned}$$

where  $G := [G_i]_{i=1, \dots, N}$ , is small in the  $C^1$ -norm as well. To this aim, note that  $f$  and its derivative  $\frac{\partial f}{\partial(\theta, r)}$  are linear in the entries  $(\kappa, \tilde{\gamma})_{ij}$ ,  $i \in \{1, \dots, N\}$ ,  $j \in \{1, \dots, 2N\}$  of the matrix  $(\kappa, \tilde{\gamma})$ . Furthermore, the coefficients multiplying  $(\kappa, \tilde{\gamma})_{ij}$ , are smooth functions of  $(\theta, r, \Phi)$  and are uniformly bounded on  $\overline{\mathcal{M}} \times \mathbb{R}^{N \times (2N+3)}$ . Similarly, for the product  $Gg$  and its derivative  $\frac{\partial(Gg)}{\partial(\theta, r)}$ . It thus follows that there exists  $C_f, C_g > 0$ ,  $C_f, C_g$  independent of  $\kappa, \tilde{\gamma}, \omega$ , and  $\Phi$ , such that

$$\|f|_{\overline{\mathcal{M}}}(\cdot, \cdot, \kappa, \tilde{\gamma}, \Phi)\|_1 \leq C_f |(\kappa, \tilde{\gamma})| \quad (18a)$$

$$\|G|_{\overline{\mathcal{M}}}(\cdot)g|_{\overline{\mathcal{M}}}(\cdot, \cdot, \kappa, \tilde{\gamma}, \Phi)\|_1 \leq C_g |(\kappa, \tilde{\gamma})|, \quad (18b)$$

that is both  $f$  and  $g$  are  $|(\kappa, \tilde{\gamma})|$ -small in the  $C^1$ -norm. Note that the constants  $C_f, C_g$  solely depend on the natural radius  $\rho_i$ .

We can finally apply [16, Theorem 4.1] to conclude the existence of  $\delta'_h > 0$ , independent of  $\kappa, \tilde{\gamma}, \omega$ , and  $\Phi$ , such that, if

$$|(\kappa, \tilde{\gamma})| \leq \delta'_h, \quad (19)$$

then (16) still has an attractive normally hyperbolic invariant N-torus  $\mathcal{T}_p \subset \overline{\mathcal{M}}$ , which is  $|(\kappa, \tilde{\gamma})|$ -near in the  $C^1$ -norm to  $\mathcal{T}_0$ . In particular, there exists  $C_h > 0$  such that, if  $|(\kappa, \tilde{\gamma})| < \delta'_h$ , then

$$|r - \rho| \leq C_h |(\kappa, \tilde{\gamma})|, \quad \forall (\theta, r) \in \mathcal{T}_p, \quad (20)$$

where again  $C_h$  is independent of  $\kappa, \tilde{\gamma}, \omega$ , and  $\Phi$ . The fact that  $\delta'_h$  and  $C_h$  depends only on the natural radius  $\rho_i$ , comes from the fact that the linearization (9) of the unperturbed dynamics (7) solely depends on  $\rho_i$ ,  $i = 1, \dots, N$ .

To prove the theorem, it remains to pick  $|(\kappa, \tilde{\gamma})|$  sufficiently small that  $\mathcal{T}_p \subset (\underline{\mathcal{M}} \setminus \partial \underline{\mathcal{M}})$ . Indeed, the compact manifold  $\underline{\mathcal{M}}$ , defined in (17), is the region where the compactified (16) and the original (6) dynamics coincide. Since normal hyperbolicity is a local concept, (16) is normally hyperbolic at a manifold  $\mathcal{T}_p \subset (\underline{\mathcal{M}} \setminus \partial \underline{\mathcal{M}})$  if and only if so it does (6). To this aim, by picking

$$\delta_h := \frac{\min_{i=1, \dots, N} \rho_i}{4C_h},$$

and

$$|(\kappa, \tilde{\gamma})| < \delta_h,$$

it follows from (19), (20), and the definition (17) of  $\underline{\mathcal{M}}$ , that for all  $(\theta, r) \in \mathcal{T}_p$ , and all  $i = 1, \dots, N$

$$|r_i - \rho_i| \leq |r - \rho| \leq C_h |(\kappa, \tilde{\gamma})| < C_h \frac{\min_{i=1, \dots, N} \rho_i}{4C_h} \leq \frac{\rho_i}{4},$$

which ensures that  $\mathcal{T}_p \subset \underline{\mathcal{M}}$ .  $\square$

#### IV. EXISTENCE OF PHASE-LOCKED SOLUTIONS IN THE PHASE DYNAMICS

Based on the analysis in Section III we formulate the following assumption, which, in view of Theorem 1 and Remark 2, is verified to the first order in the coupling and feedback strengths, provided that the small coupling condition (11) is satisfied (see Remarks 1 and 2).

**Assumption 1.** For all  $i = 1, \dots, N$ , the solution of (1) satisfies  $|z_i(t)| = \rho_i$ , for all  $t \geq 0$ .

This assumption is commonly made in synchronization studies [8], [7], [6], [20], [9]. The analysis in Section III above provides a rigorous justification to it.

In the remainder of this section, we rely on Assumption 1 and study the existence of phase-locked solutions in (13)-(14), where phase-locked solutions are defined as follows.

**Definition 2.** A solution  $\{\theta_i^*\}_{i=1,\dots,N}$  of (13) or (6) is said to be *phase-locked* if it satisfies

$$\dot{\theta}_j^*(t) - \dot{\theta}_i^*(t) = 0, \quad \forall i, j = 1, \dots, N, \quad \forall t \geq 0. \quad (21)$$

A phase-locked solution is *oscillating* if, in addition,  $\dot{\theta}_i^*(t) \neq 0$ , for almost all  $t \geq 0$  and all  $i = 1, \dots, N$ .

Phase-locking is trivially equivalent to the existence of a matrix  $\Delta := [\Delta_{ij}]_{i,j=1,\dots,N}$ , such that

$$\theta_j^*(t) - \theta_i^*(t) = \Delta_{ij}, \quad \forall i, j = 1, \dots, N, \quad \forall t \geq 0, \quad (22)$$

or to the existence of a continuous function  $\Omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  such that, for each  $i = 1, \dots, N$ ,

$$\theta_i^*(t) = \int_0^t \Omega(s) ds + \theta_i^*(0), \quad \forall t \geq 0, \quad (23)$$

Continuity of  $\Omega$  follows from the continuity of the solution of (13) (see [21, Theorem 3.1]).

The main result of [9] stated that, generically, the phase dynamics (13)-(14) admits no oscillating phase-locked solutions for the special case when  $\delta = 0$ ,  $\eta = 0$ ,  $\varphi = 0$  and  $\psi = 0$ . The rest of this section consists in extending that result to the more general case (13)-(14).

##### A. The fixed point equation

We start by identifying the phase-locked solutions or, equivalently, the fixed points of the incremental dynamics of (13), i.e.  $\dot{\theta}_i - \dot{\theta}_j = 0$ , for all  $i, j = 1, \dots, N$ . Given some

initial conditions  $\theta^*(0)$ , this fixed points equation reads

$$\begin{aligned} \omega_i &- \sum_{h=1}^N \kappa_{ih} \sin(\delta_{ih} + \eta_i) + \sum_{h=1}^N k_{ih} \sin(\Delta_{ih} + \delta_{ih} + \eta_h) \\ &+ \sum_{h=1}^N \gamma_{ih} \frac{\sin \varphi_h + \cos \varphi_h}{2} \sin(\Delta_{ih} + \phi_{ih} + \psi_h) \\ &- \omega_j + \sum_{h=1}^N \kappa_{jh} \sin(\delta_{jh} + \eta_j) - \sum_{h=1}^N k_{jh} \sin(\Delta_{jh} + \delta_{jh} + \eta_h) \\ &- \sum_{h=1}^N \gamma_{jh} \frac{\sin \varphi_h + \cos \varphi_h}{2} \sin(\Delta_{jh} + \phi_{jh} + \psi_h) \\ &+ \sum_{h=1}^N \frac{\sin \varphi_h - \cos \varphi_h}{2} \gamma_{ih} \sin(2\Lambda_\Omega(t) + \Delta_{ih} + 2\theta_i^*(0) - \phi_{ih} + \psi_h) \\ &- \sum_{h=1}^N \frac{\sin \varphi_h - \cos \varphi_h}{2} \gamma_{jh} \sin(2\Lambda_\Omega(t) + \Delta_{jh} + 2\theta_j^*(0) - \phi_{jh} + \psi_h) \\ &= 0. \end{aligned} \quad (24)$$

where the phase differences  $\Delta \in \mathbb{R}^{N \times N}$  and the common frequency of oscillation  $\Omega : \mathbb{R} \rightarrow \mathbb{R}$  are defined in (22) and (23) respectively.

Let us introduce some notation. The fixed point equation (24) must be solved in  $\Delta$  and  $\Omega$ . It is parametrized, apart from the natural frequencies  $\omega$ , by the elements of the matrix  $\Upsilon \in \mathbb{R}^{N \times (4N+3)}$ , which is defined as

$$\Upsilon := (\kappa, \tilde{\gamma}, \Phi), \quad (25)$$

where  $\Phi$  is defined in (3). Let us denote the first four (time-independent) lines of (24) as the function  $\Phi_{ij}^{TI} : \mathbb{R}^N \times \mathbb{R}^{N \times (4N+3)} \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ , that is

$$\Phi_{ij}^{TI}(\omega, \Upsilon, \Delta) := \quad (26)$$

$$\begin{aligned} \omega_i &- \sum_{h=1}^N \kappa_{ih} \sin(\delta_{ih} + \eta_i) + \sum_{h=1}^N k_{ih} \sin(\Delta_{ih} + \delta_{ih} + \eta_h) \\ &+ \sum_{h=1}^N \gamma_{ih} \frac{\sin \varphi_h + \cos \varphi_h}{2} \sin(\Delta_{ih} + \phi_{ih} + \psi_h) \\ &- \omega_j + \sum_{h=1}^N \kappa_{jh} \sin(\delta_{jh} + \eta_j) - \sum_{h=1}^N k_{jh} \sin(\Delta_{jh} + \delta_{jh} + \eta_h) \\ &- \sum_{h=1}^N \gamma_{jh} \frac{\sin \varphi_h + \cos \varphi_h}{2} \sin(\Delta_{jh} + \phi_{jh} + \psi_h). \end{aligned}$$

Similarly we denote the last two (time-dependent) lines of (24) as the function  $\Phi_{ij}^{TD} : \mathbb{R} \times \mathbb{R}^{N \times (4N+3)} \times \mathbb{R}^{N \times N} \times \mathbb{R}^N \rightarrow \mathbb{R}$ , that is

$$\Phi_{ij}^{TD}(t, \Upsilon, \Delta, \theta^*(0)) := \quad (27)$$

$$\begin{aligned} &+ \sum_{h=1}^N \frac{\sin \varphi_h - \cos \varphi_h}{2} \gamma_{ih} \sin(2\Lambda_\Omega(t) + \Delta_{ih} + 2\theta_i^*(0) - \phi_{ih} + \psi_h) \\ &- \sum_{h=1}^N \frac{\sin \varphi_h - \cos \varphi_h}{2} \gamma_{jh} \sin(2\Lambda_\Omega(t) + \Delta_{jh} + 2\theta_j^*(0) - \phi_{jh} + \psi_h). \end{aligned}$$

The following lemma states that the problem of finding an oscillating phase-locked solution to (13)-(14) can be reduced to solving a set of nonlinear algebraic equations in terms of the phase differences  $\Delta$  and the collective frequency of oscillation  $\Omega$ .

**Lemma 2.** *For all initial conditions  $\theta^*(0) \in T^N$ , all natural frequencies  $\omega \in \mathbb{R}^N$ , all parameters  $\Upsilon \in \mathbb{R}^{N \times (4N+3)}$ , if system (13) admits an oscillating phase-locked solution starting in  $\theta^*(0)$  with phase differences  $\Delta$  and collective frequency of oscillation  $\Omega$ , then, for all  $1 \leq i < j \leq N$ , the functions defined in (26) and (27) satisfy*

$$\Phi_{ij}^{TI}(\omega, \Upsilon, \Delta) = 0, \quad (28a)$$

$$\Phi_{ij}^{TD}(t, \Upsilon, \Delta, \theta^*(0)) = 0. \quad (28b)$$

*Proof.* The proof follows along the same lines as those of [9, Lemma 1]. Firstly, note that, by definition, the fixed point equation (24) can be rewritten as

$$\Phi_{ij}^{TI}(\omega, \Upsilon, \Delta) + \Phi_{ij}^{TD}(t, \Upsilon, \Delta, \theta^*(0)) = 0.$$

Since  $\Phi_{ij}^{TI}(\omega, \Upsilon, \Delta)$  is constant, this is equivalent to writing

$$\Phi_{ij}^{TI}(\omega, \Upsilon, \Delta) = b_{ij},$$

$$\Phi_{ij}^{TD}(t, \Upsilon, \Delta, \theta^*(0)) = -b_{ij}, \quad (29)$$

for some constant  $b_{ij} \in \mathbb{R}$ . We claim that, if the phase-locked solution is oscillating, then necessarily  $b_{ij} = 0$ . To see this, differentiate (29) with respect to time. We obtain, for all  $t \geq 0$ ,

$$\begin{aligned} & \left\{ \sum_{h=1}^N \gamma_{ih} \left[ \cos \psi_h \cos(2\Lambda_\Omega(t) + \Delta_{ih} - \phi_i + 2\theta_i^*(0)) \right. \right. \\ & \quad \left. \left. - \sin \psi_h \sin(2\Lambda_\Omega(t) + \Delta_{ih} - \phi_i + 2\theta_i^*(0)) \right] \right. \\ & \quad \left. - \sum_{h=1}^N \gamma_{jh} \left[ \cos \psi_h \cos(2\Lambda_\Omega(t) + \Delta_{jh} - \phi_j + 2\theta_j^*(0)) \right. \right. \\ & \quad \left. \left. - \sin \psi_h \sin(2\Lambda_\Omega(t) + \Delta_{jh} - \phi_j + 2\theta_j^*(0)) \right] \right\} \\ & \quad \times 2\Omega(t) = 0. \end{aligned} \quad (30)$$

Since the solution is oscillating,  $\Omega$  is a non-identically zero continuous function, and, thus, there exists an open interval  $(\underline{t}, \bar{t})$ , such that  $\Omega(t) \neq 0$ , for all  $t \in (\underline{t}, \bar{t})$ . Hence, (30) implies that

$$\begin{aligned} & \sum_{h=1}^N \gamma_{ih} \left[ \cos \psi_h \cos(2\Lambda_\Omega(t) + \Delta_{ih} - \phi_i + 2\theta_i^*(0)) \right. \\ & \quad \left. - \sin \psi_h \sin(2\Lambda_\Omega(t) + \Delta_{ih} - \phi_i + 2\theta_i^*(0)) \right] \\ & \quad - \sum_{h=1}^N \gamma_{jh} \left[ \cos \psi_h \cos(2\Lambda_\Omega(t) + \Delta_{jh} - \phi_j + 2\theta_j^*(0)) \right. \\ & \quad \left. - \sin \psi_h \sin(2\Lambda_\Omega(t) + \Delta_{jh} - \phi_j + 2\theta_j^*(0)) \right] = 0, \end{aligned} \quad (31)$$

for all  $t \in (\underline{t}, \bar{t})$ . By differentiating (31) with respect to time and considering once again that  $\Omega(t) \neq 0$  for all  $t \in (\underline{t}, \bar{t})$ , one gets

$$\Phi_{ij}^{TD}(t, \Upsilon, \Delta, \theta^*(0)) = 0$$

for all  $t \in (\underline{t}, \bar{t})$ , that is, at the light of (29),  $b_{ij} = 0$ , which concludes the proof.  $\square$

### B. Invertibility of the time-independent part of the fixed point equation

In the following lemma we show that, for a generic choice of the parameters, the time-independent part (28a) of the fixed point equation (24) can be inverted to give the phase differences  $\Delta$  in term of the natural frequencies  $\omega$  and the parameter matrix  $\Upsilon$ .

**Lemma 3.** *There exists a set  $\mathcal{N} \subset \mathbb{R}^N \times \mathbb{R}^{N \times (4N+3)}$ , and a set  $\mathcal{N}_0 \subset \mathcal{N}$  satifying  $\mu(\mathcal{N}_0) = 0$ , such that (28a) with natural frequencies  $\omega^* \in \mathbb{R}^N$  and parameters  $\Upsilon^* \in \mathbb{R}^{N \times (4N+3)}$  admits a solution  $\Delta^* \in \mathbb{R}^{N \times N}$  if and only if  $(\omega^*, \Upsilon^*) \in \mathcal{N}$ . Moreover, for all  $(\omega^*, \Upsilon^*) \in \mathcal{N} \setminus \mathcal{N}_0$ , there exists a neighborhood  $U$  of  $(\omega^*, \Upsilon^*)$ , a neighborhood  $W$  of  $\Delta^*$ , and an analytic function  $f : U \rightarrow W$ , such that, for all  $(\omega, \Upsilon) \in U$ ,  $(\omega, \Upsilon, \Delta := f(\omega, \Upsilon))$  is the unique solution of (28a) in  $U \times W$ .*

**Remark 3.** In this generalized version, we prove the analyticity of  $f$ , instead of simply smoothness as for [9, Lemma 2], since this permits to largely simplify the proof of the existence Theorem 2 below.

*Proof. (SKETCH)* The first part of the proof follows exactly the same steps as [9, Lemma 2], with the matrix  $\Upsilon$  at the place of the matrix  $\Gamma$ , and with the two function  $F$  and  $\hat{F}$  redefined as follows. By letting  $y_i := \Delta_{iN}$ ,  $i = 1, \dots, N-1$ , we let

$$F_i(\omega, \Upsilon, y) := \Phi_{iN}^{TI}(\omega, \Upsilon, \Delta(y)),$$

and

$$\hat{F}_i(\hat{\omega}_i, \Upsilon, y) := \Phi_{iN}^{TI}(0, \Upsilon, \Delta(y)) + \hat{\omega}_i,$$

where  $\Delta_{nN}(y) := y_n$ ,  $\Delta_{nm}(y) = y_m - y_n$ , and  $\hat{\omega}_m := \omega_m - \omega_N$ ,  $n = 1, \dots, N$ ,  $m = 1, \dots, N-1$ . The end of the proof is slightly different since, in order to prove the analyticity of  $f$ , instead of just smoothness as in [9, Lemma 2], one has to invoke the fact that  $F$  is analytic and then apply the analytic implicit function theorem [22, Theorem 2.3.5]. For more details, we invite the reader to retrace the proof of [9, Lemma 2], with the above modifications in mind.  $\square$

### C. Non-existence of oscillating phase-locked solutions

In the following theorem, we show that, for a generic choice of the parameters, no oscillating phase-locked solution exists in the phase dynamics (13).

**Theorem 2.** *For all initial conditions  $\theta^*(0) \in T^N$ , and for almost all  $\omega \in \mathbb{R}^N$  and  $\Upsilon \in \mathbb{R}^{N \times (4N+3)}$ , as defined in (25), (13)-(14) admits no oscillating phase-locked solution starting in  $\theta^*(0)$ .*



*Proof.* Observe that, if  $(\omega, \Upsilon) \notin \mathcal{N}$ , then, by Lemma 3, the time-independent part of the fixed point equation (28a) admits no solutions and, thus, by Lemma 2, the phase dynamics (13) admits no oscillating phase-locked solution.

Therefore, let us assume that  $(\omega, \Upsilon) \in \mathcal{N}$ . We claim that there exists  $\mathcal{M}_0 \subset \mathcal{N}$ , with  $\mu(\mathcal{M}_0) = 0$ , such that, given initial conditions  $\theta^*(0)$ , if there exists an oscillating phase-locked solution of (13) starting in  $\theta^*(0)$ , then  $(\omega, \Upsilon) \in \mathcal{N}_0 \cup \mathcal{M}_0$ , where  $\mathcal{N}_0$  is defined in Lemma 3 in Section IV-B. If our claim holds true, noticing that  $\mu(\mathcal{M}_0 \cup \mathcal{N}_0) = 0$ , then the theorem is proved.

We want to construct  $\mathcal{M}_0$  as the zeros of a suitable analytic function, thus ensuring that it has zero Lebesgue measure [22, Page 83].

Given  $(\omega, \Upsilon) \in \mathcal{N} \setminus \mathcal{N}_0$ , it follows from Lemma 3 that there exists a unique  $\Delta(\omega, \Upsilon)$  such that  $(\omega, \Upsilon, \Delta(\omega, \Upsilon))$  is solution to (28a). That is the function

$$\begin{aligned} \mathcal{N} \setminus \mathcal{N}_0 &\longrightarrow \mathbb{R}^{N \times N} \\ (\omega, \Upsilon) &\longmapsto \Delta(\omega, \Upsilon) \end{aligned}$$

is well defined. It is also analytic, since, again by Lemma 3, for all  $(\omega, \Upsilon) \in \mathcal{N} \setminus \mathcal{N}_0$  it is analytic in a neighborhood  $U$  of  $(\omega, \Upsilon)$ . Given a pair of indexes  $i \neq j$ , consider the function  $\tilde{\Phi}_{ij}^{TD}$  defined as

$$\tilde{\Phi}_{ij}^{TD} : \begin{cases} \mathcal{N} \setminus \mathcal{N}_0 \rightarrow \mathbb{R} \\ (\omega, \Upsilon) \mapsto \Phi_{ij}^{TD}(0, \Upsilon, \Delta(\omega, \Upsilon), \theta^*(0)) \end{cases} \quad (32)$$

where  $\Phi_{ij}^{TD}$  is defined in (27). The function  $\tilde{\Phi}_{ij}^{TD}$  is analytic on  $\mathcal{N} \setminus \mathcal{N}_0$ , since it is the composition of two analytic functions [22, Proposition 1.4.2]. We define  $\mathcal{M}_0$  as the zero set of  $\tilde{\Phi}_{ij}^{TD}$ , that is

$$\mathcal{M}_0 := \{(\omega, \Upsilon) \in \mathcal{N} \setminus \mathcal{N}_0 : \tilde{\Phi}_{ij}^{TD}(\omega, \Upsilon) = 0\}. \quad (33)$$

As anticipated above, since  $\tilde{\Phi}_{ij}^{TD}$  is analytic,  $\mu(\mathcal{M}_0) = 0$ . By construction, if  $(\omega, \Upsilon) \in (\mathcal{N} \setminus \mathcal{N}_0) \setminus \mathcal{M}_0$ , then  $\tilde{\Phi}_{ij}^{TD}(\omega, \Upsilon) \neq 0$ , that is, by the definition of  $\tilde{\Phi}_{ij}^{TD}$  in (32), the time-dependent part of the fixed point equation (28b) admits no solutions. By Lemma 2, this implies that, if there exists an oscillating phase-locked solution starting from  $\theta^*(0)$ , then necessarily

$$\begin{aligned} (\omega, \Upsilon) &\in \mathcal{N} \setminus [(\mathcal{N} \setminus \mathcal{N}_0) \setminus \mathcal{M}_0] \\ &= \mathcal{N} \setminus [\mathcal{N} \setminus (\mathcal{N}_0 \cup \mathcal{M}_0)] \\ &= \mathcal{N}_0 \cup \mathcal{M}_0, \end{aligned}$$

which proves the claim.  $\square$

## V. EXISTENCE OF PHASE-LOCKED SOLUTIONS IN THE ORIGINAL DYNAMICS

We can readily apply Theorem 2 and the perturbation analysis of Section III to study the existence of oscillating phase-locked solutions in the full dynamics (1).

**Corollary 1.** *For all  $\theta(0) \in T^N$ , for almost all  $\omega^\circ \in [-1, 1]^N$ , almost all  $\kappa^\circ, \tilde{\gamma}^\circ \in [-1, 1]^{N \times N} \setminus \{0\}$ , and almost*

*all  $\Phi \in \mathbb{R}^{N \times (2N+3)}$ , such that*

$$\max_{i=1, \dots, N} |\omega_i^\circ| - \sum_{j=1}^N \left( |\kappa_{ij}^\circ| + \frac{|\kappa_{ij}^\circ| \rho_j}{\rho_i} + 2 \frac{|\tilde{\gamma}_{ij}^\circ| \rho_j}{\rho_i} \right) > 0 \quad (34)$$

*there exists  $\bar{\varepsilon} > 0$  such that, for all  $\varepsilon \in (0, \bar{\varepsilon}]$ , system (6) with natural frequencies  $\omega = \varepsilon \omega^\circ$ , coupling matrix  $\kappa = \varepsilon \kappa^\circ$ , and feedback gain  $\tilde{\gamma} = \varepsilon \tilde{\gamma}^\circ$ , is normally hyperbolic at an invariant manifold  $\mathcal{T}_p \subset T^N \times \mathbb{R}_{>0}^N$  such that no oscillating phase-locked solution exists starting in  $(\theta(0), r(0)) \in \mathcal{T}_p$ .*

Corollary 1 states that, for almost all natural frequency dispersion (i.e.  $\omega^\circ$ ) and coupling and feedback topologies (i.e.  $\kappa^\circ, \tilde{\gamma}^\circ$ ), the full dynamics (1) admits no oscillating phase-locked solutions on the attractive normally hyperbolic invariant manifold  $\mathcal{T}_p$ , provided that the absolute magnitude of the natural frequencies and the coupling and feedback strengths are small (i.e.  $\varepsilon \leq \bar{\varepsilon}$ ) and that the reduced phase dynamics with the same parameters is oscillating, i.e.  $\theta \neq 0$ , as implied by (34).

*Proof.* We need some notation to distinguish the solutions of the full (6) and reduced (14) phase dynamics. Given  $(\theta(0), r(0)) \in T^N \times \mathbb{R}_{>0}^N$ , we thus let  $\theta^0(\cdot, \theta(0); \omega, \kappa, \tilde{\gamma}, \Phi)$  denote the solution of (14) with parameters  $(\omega, \kappa, \tilde{\gamma}, \Phi)$ , whereas we let  $\theta(\cdot, (\theta(0), r(0)); \omega, \kappa, \tilde{\gamma}, \Phi)$  denote the solution of the full dynamics (6) with the same set of parameters.

It follows from Theorem 2 that for all  $\theta(0) \in T^N$ , and for almost all  $\omega^\circ \in [-1, 1]^N$ ,  $(\kappa^\circ, \tilde{\gamma}^\circ) \in \mathbb{R}^{N \times 2N} \setminus \{0\}$  and  $\Phi \in \mathbb{R}^{N \times (2N+3)}$ , the reduced phase dynamics (14) with natural frequencies  $\omega = \omega^\circ$ , coupling and feedback gain  $\kappa = \kappa^\circ, \tilde{\gamma} = \tilde{\gamma}^\circ$ , and phases  $\Phi$  admits no oscillating phase-locked solutions. Since, from (34) and (13)-(14),  $\theta^0$  is oscillating, there thus exist  $i, j \in 1, \dots, N$ ,  $i \neq j$ , and  $C_1 > 0$  such that

$$\sup_{t \in \mathbb{R}} |\dot{\theta}_j^0(t, \theta(0); \omega^\circ, \kappa^\circ, \tilde{\gamma}^\circ, \Phi) - \dot{\theta}_i^0(t, \theta(0); \omega^\circ, \kappa^\circ, \tilde{\gamma}^\circ, \Phi)| = C_1. \quad (35)$$

Since  $|\dot{\theta}_j^0(t) - \dot{\theta}_i^0(t)|$  is linear in  $\omega, \kappa$ , and  $\tilde{\gamma}$ , (35) implies that, given  $\varepsilon > 0$

$$\sup_{t \in \mathbb{R}} |\dot{\theta}_j^0(t, \theta(0); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi) - \dot{\theta}_i^0(t, \theta(0); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi)| = \varepsilon C_1. \quad (36)$$

$$-\dot{\theta}_i^0(t, \theta(0); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi) \Big| = \varepsilon C_1.$$

The perturbation analysis of Section III, summarized in Remark 2, implies that the difference between the full and reduced phase dynamics is of the same order as  $|(\kappa, \tilde{\gamma})|^2$ , provided that the small coupling condition (11) is satisfied. That is, if  $\varepsilon |(\kappa^\circ, \tilde{\gamma}^\circ)| < \delta_h$  or, equivalently, considering  $(\kappa^\circ, \tilde{\gamma}^\circ) \neq 0$ ,

$$\varepsilon < \frac{\delta_h}{|(\kappa^\circ, \tilde{\gamma}^\circ)|}, \quad (37)$$

then there exists a manifold  $\mathcal{T}_p \subset T^N \times \mathbb{R}_{>0}^N$ , normally hyperbolic and invariant for (6), and a constant  $C_2 > 0$  such that, for all  $(\theta(0), r(0)) \in \mathcal{T}_p$  and all  $l = 1, \dots, N$ ,

$$\sup_{t \in \mathbb{R}} |\dot{\theta}_l(t, (\theta(0), r(0)); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi)| \quad (38)$$

$$-\dot{\theta}_l^0(t, \theta(0); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi) \Big| \leq C_2 \varepsilon^2 |(\kappa^\circ, \tilde{\gamma}^\circ)|^2.$$

Let us denote, for all  $l = 1, \dots, N$ ,

$$\dot{\theta}_l(t, \varepsilon) := \dot{\theta}_l(t, (\theta(0), r(0)); \varepsilon \omega^\circ, \varepsilon \kappa^\circ, \varepsilon \tilde{\gamma}^\circ, \Phi)$$

and similarly for  $\theta^0$ . By using (36) and (38), it follows that, if (37) is satisfied, then

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left| \dot{\theta}_i(t, \varepsilon) - \dot{\theta}_j(t, \varepsilon) \right| \\ &= \sup_{t \in \mathbb{R}} \left| \dot{\theta}_i(t, \varepsilon) - \dot{\theta}_i^0(t, \varepsilon) - \dot{\theta}_j(t, \varepsilon) + \dot{\theta}_j^0(t, \varepsilon) \right. \\ & \quad \left. + \dot{\theta}_i^0(t, \varepsilon) - \dot{\theta}_j^0(t, \varepsilon) \right| \\ &\geq \sup_{t \in \mathbb{R}} \left| \dot{\theta}_i^0(t, \varepsilon) - \dot{\theta}_j^0(t, \varepsilon) \right| - \sup_{t \in \mathbb{R}} \left| \dot{\theta}_i(t, \varepsilon) - \dot{\theta}_i^0(t, \varepsilon) \right| \\ & \quad - \sup_{t \in \mathbb{R}} \left| \dot{\theta}_j(t, \varepsilon) - \dot{\theta}_j^0(t, \varepsilon) \right| \\ &\geq C_1 \varepsilon - 2C_2 \varepsilon^2 |(\kappa^\circ, \tilde{\gamma}^\circ)|^2 \end{aligned}$$

Thus, if

$$\varepsilon < \min \left\{ \frac{\delta_h}{|(\kappa^\circ, \tilde{\gamma}^\circ)|}, \frac{C_1}{2C_2 |(\kappa^\circ, \tilde{\gamma}^\circ)|^2} \right\} =: \bar{\varepsilon}$$

then  $\sup_{t \in \mathbb{R}} |\dot{\theta}_i(t, \varepsilon) - \dot{\theta}_j(t, \varepsilon)| > 0$ , that is (1) admits no oscillating phase-locked solutions starting in  $(\theta(0), r(0))$ .  $\square$

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